CUNTZ-KRIEGER ALGEBRAS AND A GENERALIZATION OF CATALAN NUMBERS

KENGO MATSUMOTO

Abstract. We first observe that the relations of the canonical generating isometries of the Cuntz algebra $\mathcal{O}_N$ are naturally related to the N-colored Catalan numbers. For a directed graph G, we generalize the Catalan numbers by using the canonical generating partial isometries of the Cuntz-Krieger algebra $\mathcal{O}_{A^G}$ for the transition matrix A^G of G. The generalized Catalan numbers $c_n^G, n = 0, 1, 2, \ldots$ enumerate the number of Dyck paths and oriented rooted trees for the graph G. Its generating functions will be studied.

Keywords: Catalan numbers, directed graphs, Dyck path, oriented rooted trees, C^*-algebras, Cuntz-Krieger algebras, generating functions,
Mathematics Subject Classification 2000: Primary 05A15; Secondary 46L05.

1. Introduction

In the theory of combinatorics, the process of enumerating objects of various nature has been considered (cf. [26], [27], etc.). It yields a sequence of positive integers and its generating function. The Catalan numbers are one of typical examples of such sequences. The numbers enumerate various objects, brackets, Dyck paths, rooted trees, triangulation of polygons, etc. (cf. [4], [18], etc.).

In this paper, we will first observe that the operator relations of the canonical generating isometries of the Cuntz algebra $\mathcal{O}_N$ are naturally related to the N-colored Catalan numbers $c_n^{(N)}, n = 0, 1, 2, \ldots$. The Cuntz-Krieger algebras are natural generalization of the Cuntz algebras from the view point of topological Markov shifts. They are defined by a directed graph and have generating partial isometries satisying certain operator relations coming from the structure of the graph. For a directed graph G, we generalize the Catalan numbers by using the canonical generating partial isometries of the Cuntz-Krieger algebra $\mathcal{O}_{A^G}$ for the transition matrix A^G of G. We call the generalized Catalan numbers $c_n^G, n = 0, 1, 2, \ldots$ G-Catalan numbers. We will then show that the generalized Catalan numbers enumerate the Dyck paths and the oriented rooted trees associated to the graph G. Let $\{v_1, \ldots, v_N\}$ be the vertex set of G. The generalized Catalan numbers $c_n^G, n = 0, 1, 2, \ldots$ are of the form

$$c_n^G = \sum_{i=1}^{N} c_n^G(i), \qquad n = 0, 1, 2, \ldots$$

where $c_n^G(i)$ enumerate the numbers rooted at the vertex v_i, and $c_0^G(i)$ is defined to be 1 for $i = 1, \ldots, N$ so that $c_0^G = N$. They satisfy the following relation:

$$c_{n+1}^G(i) = \sum_{k=0}^{n} c_{n-k}^G(i) \sum_{j=1}^{N} A_G(j,i) c_k^G(j) \tag{1.1}$$

where $A_G(j,i)$ for $v_i, v_j \in V$ denotes the number of directed edges from v_j to v_i in the graph G. Let $f^G(x)$ be the generating function for the G-Catalan numbers $c_n^G, n = 0, 1, \ldots$. It is defined by

$$f^G(x) = \sum_{n=0}^{\infty} c_n^G x^n. \tag{1.2}$$

Let $f_i^G(x)$ be the generating function for the sequence $c_n^G(i), n = 0, 1, \ldots$. The functions satisfy the following equations

$$f^G(x) = \sum_{i=1}^{N} f_i^G(x), \tag{1.3}$$

$$f_i^G(x) = 1 + x f_i^G(x) \sum_{j=1}^{N} A_G(j,i) f_j^G(x) \qquad \text{for } i = 1, \ldots, N. \tag{1.4}$$

We will prove that the family $f_i^G(x), i = 1, \ldots, N$ of functions is uniquely determined by the above relations by the implicit function theorem (Theorem 6.5). The radius of convergence of $f_i^G(x)$ does not depend on $i = 1, \ldots, N$, and is determined algebraically as a solution of an eigenvalue problem of a certain matrix associated with the transition matrix A_G of the graph G (Theorem 7.2). We will prove a formula of $c_n^G(i)$ by using the matrix A_G. Put

$$F_i(w_1, \ldots, w_N) = (w_i + 1) \sum_{j=1}^{N} A_G(j,i)(w_j + 1),$$
$$F_i^n(w_1, \ldots, w_N) = F_i(w_1, \ldots, w_N)^n, \qquad i = 1, \ldots, N.$$

We will show the following integral formulae

$$c_n^G(i) = \frac{1}{2\pi n\sqrt{-1}} \int_C \frac{F_j^n(w_1, \ldots, w_N)}{w_j^n} dw_i, \qquad i, j = 1, \ldots, N$$

hold, where the above integral is a contour integral along a closed curve C around the origin (Theorem 8.1). By using the above formulae, we will compute some examples. In particular, the G-Catalan numbers for the graph G bellow are computed as

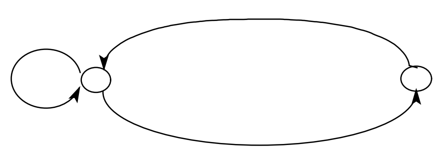

FIGURE 1

$$c_n^G = \frac{2}{n}\binom{3n+1}{n-1} + \frac{1}{n}\binom{3n}{n-1} = \frac{2}{n+1}\binom{3n}{n}.$$

Hence
$$c_0^G = 2, \quad c_1^G = 3, \quad c_2^G = 10, \quad c_3^G = 42, \quad c_4^G = 198, \quad \dots .$$
This sequence is regarded to be the Fibonacci version of the Catalan numbers. The radius of convergence of the function $f^G(x)$ is computed to be $\frac{4}{27}$.

We will also define generalized Catalan numbers $c_n^{G,\varphi}$ by using KMS state φ on the C^*-algebra $\mathcal{O}_{A^G}$ for the gauge action. The numbers $c_n^{G,\varphi}$ are computed by the G-Catalan numbers with the Perron-Frobenius eigenvalue and its eigenvector for the matrix A^G (Proposition 9.3).

2. The Cuntz algebras and Catalan numbers

Throughout this section N is a fixed positive integer greater than 1. For a set Ω, we denote by $|\Omega|$ the number of Ω. Let $S_1, \dots, S_N$ be a family of bounded linear operators on a Hilbert space satisfying the following condition
$$\sum_{j=1}^N S_j S_j^* = 1, \qquad S_i^* S_i = 1 \quad \text{for} \quad i = 1, \dots, N \tag{2.1}$$
Let $\mathcal{O}_N$ be the C^*-algebra generated by the family $S_1, \dots, S_N$. The algebra $\mathcal{O}_N$ is called the Cuntz algebra ([2]). It is well-known that the algebraic structure of $\mathcal{O}_N$ does not depend on the choice of the family $S_1, \dots, S_N$ satisfying the relations (2.1). For a word $\mu = (\mu_1, \dots, \mu_n)$ of $\{1, \dots, N\}$, we put
$$S_\mu = S_{\mu_1} \cdots S_{\mu_n} \quad \text{and} \quad S_\mu^* = S_{\mu_n}^* \cdots S_{\mu_1}^*.$$
Put $\Sigma_N = \{S_1^*, \dots, S_N^*, S_1, \dots, S_N\}$. Let W_{2n}^N be the set of all words of Σ_N of length $2n$:
$$W_{2n}^N = \{(X_1, \dots, X_{2n}) \mid X_i \in \Sigma_N, \, i = 1, \dots, 2n\}.$$
For $X = (X_1, \dots, X_{2n}) \in W_{2n}^N$, define $\pi_N(X)$ to be the element $X_1 \cdots X_{2n}$ in $\mathcal{O}_N$. By [2, 1.3 Lemma], every $\pi_N(X)$ for a word X in W_{2n}^N is one of the forms:
$$0, \qquad 1, \qquad S_\mu S_\nu^* \quad \text{for some words } \mu, \nu \text{ of } \{1, \dots, N\}.$$
We set
$$B_n^N = \{X \in W_{2n}^N \mid \pi_N(X) = 1\}.$$
Define the numbers
$$c_0^{(N)} = N, \qquad c_n^{(N)} = |B_n^N| \quad \text{for } n = 1, 2, \dots .$$
The following proposition is a key in our further discussions

Lemma 2.1. *Let $X = (X_1, \dots, X_{2n})$ be a word in B_n^N.*

(i) *The number of $S_i^*, i = 1, \dots, N$ and $S_i, i = 1, \dots, N$ in X is the same.*

(ii) *The number of $S_i^*, i = 1, \dots, N$ in any starting segment from the leftmost of X is not less than the number of $S_i, i = 1, \dots, N$ in the same segment.*

Proof. (i) is clear.

(ii) Let μ, ν be the word of $\{1, \dots, N\}$ with same length. Then (2.1) implies
$$S_\mu^* S_\nu = \begin{cases} 1 & \text{if } \mu = \nu, \\ 0 & \text{if } \mu \neq \nu. \end{cases}$$
Hence, if the number of $S_i^*, i = 1, \dots, N$ in some starting segment from leftmost of X is less than the number of $S_i, i = 1, \dots, N$ in the same segment. The operator $\pi_N(X)$ is reduced to the word of the form $S_j \pi_N(Y)$ for some $j = 1, \dots, N$ and

some word Y in Σ_N. The word $S_j\pi_N(Y)$ however is not abe to be 1 for any word Y. Thus the assertion holds. $\square$

Let c_n be the usual Catalan number $\frac{1}{n+1}\binom{2n}{n}$. The above two properties in Lemma 2.1 characterize the regular bracket structures ([18, p.26]), so that we have

Proposition 2.2.

$$c_n^{(N)} = N^n \times c_n, \qquad n = 0, 1, \dots.$$

The numbers $N^n \times c_n$ are called the N-colored Catalan numbers.

Proof. Let $(_1, \dots, (_N,)_1, \dots,)_N$ be the N pairs of $2N$ brackets. Consider the following correspondence between the brackets $(_1, \dots, (_N,)_1, \dots,)_N$ and the operators $S_1^*, \dots, S_N^*, S_1, \dots, S_N$ such as

$$(_i \rightarrow S_i^*, \qquad)_i \rightarrow S_i \qquad \text{for } i = 1, \dots, N.$$

One easily sees that the set of regular brackets exactly corresponds to the set of words of B_n^N. $\square$

For example, the word

$$S_{f_4}^* S_{f_3}^* S_{f_3} S_{f_2}^* S_{f_1}^* S_{f_1} S_{f_2} S_{f_4} \cdot S_{f_3}^* S_{f_2}^* S_{f_2} S_{f_3}$$

corresponds to the word of brackets

$$(_{f_4}(_{f_3})_{f_3}(_{f_2}(_{f_1})_{f_1})_{f_2})_{f_4} \cdot (_{f_3}(_{f_2})_{f_2})_{f_3}$$

for $f_1, f_2, f_3, f_4 \in \{1, \dots, N\}$. We note that for every word $X = X_1, \dots X_{2n}$ of B_n^N, the leftmost symbol X_1 must be S_i^* for some i and the rightmost symbol X_{2n} must be S_j for some j. We denote i by $l(X)$ and j by $r(X)$ respectively.

3. The Cuntz-Krieger algebras and generalized Catalan numbers

Let $G = (V, E)$ be a directed finite graph with vertex set V and edge set E. Consider the associated edge matrix A^G for G defined by for $e, f \in E$

$$A^G(e, f) = \begin{cases} 1 & \text{if } t(e) = s(f), \\ 0 & \text{otherwise} \end{cases}$$

where $t(e), s(f)$ denote the terminal vertex of edge e and the source vertex of edge f. We henceforth assume that every vertex of G has an incoming edge and an outgoing edge so that A^G has no zero rows or columns. Consider the Cuntz-Krieger algebra $\mathcal{O}_{A^G}$ for the matrix A^G that is the universal C^*-algebra generated by partial isometries $S_e, e \in E$ subject to the following relations:

$$\sum_{f \in E} S_f S_f^* = 1, \qquad S_e^* S_e = \sum_{f \in E} A^G(e, f) S_f S_f^* \quad \text{for } e \in E. \tag{3.1}$$

For a vertex $v \in V$ we put

$$P_v = \sum_{f \in E, v = s(f)} S_f S_f^*.$$

By the above relations, we know that

(i) $P_v P_u = 0$ if $u \neq v$ for $u, v \in V$,
(ii) $P_{t(e)} = S_e^* S_e$ for $e \in E$.

The algebra $\mathcal{O}_{A^G}$ is often called a graph algebra and denoted by $\mathcal{O}_G$ [16, 17].

Let $\{e_1, \dots, e_{|E|}\}$ be the edge set E. Put $\Sigma_G = \{S_{e_1}^*, \dots, S_{e_{|E|}}^*, S_{e_1}, \dots, S_{e_{|E|}}\}$. We denote by W_{2n}^G the set of all words of Σ_G of length $2n$. For a word $X = (X_1, \dots, X_{2n}) \in W_{2n}^G$, define $\pi_G(X) = X_1 \cdots X_{2n} \in \mathcal{O}_{A^G}$ as an element of the algebra $\mathcal{O}_{A^G}$. Let $S_1, \dots, S_{|E|}$ be the canonical generating isometries of the Cuntz-algebra $\mathcal{O}_{|E|}$ satisfying (2.1). Put $\Sigma_{|E|} = \{S_1^*, \dots, S_{|E|}^*, S_1, \dots, S_{|E|}\}$ and define the correspondence

$$\Phi_G : \Sigma_G \longrightarrow \Sigma_{|E|}$$

by setting

$$\Phi_G(S_{e_i}^*) = S_i^*, \qquad \Phi_G(S_{e_i}) = S_i, \qquad i = 1, \dots, |E|.$$

The correspondence Φ_G is naturally extended to words of Σ_G. We set

$$B_n^G = \{X \in W_{2n}^G \mid \pi_G(X) \neq 0, \Phi_G(X) \in B_n^{|E|}\}.$$

A word $X = (X_1, \dots, X_{2n})$ of B_n^G is called a *G-Catalan word*. Hence a word $X = (X_1, \dots, X_{2n})$ of Σ_G is a G-Catalan word if and only if $X_1 \cdots X_{2n} \neq 0$ in $\mathcal{O}_{A^G}$ and $\Phi_G(X_1) \cdots \Phi_G(X_{2n}) = 1$ in $\mathcal{O}_{|E|}$. We set

$$c_0^G = N, \qquad c_n^G = |B_n^G| \quad \text{for } n = 1, 2, \dots$$

where N denotes the number $|V|$ of the vertex set V. We call the sequaence $c_n^G, n = 0, 1, \dots$ *the generalized Catalan number associated with the graph* G, or *G-Catalan numbers* for brevity.

Let Λ_G be the topological Markov shift

$$\Lambda_G = \{(f_i)_{i \in \mathbb{Z}} \in E^{\mathbb{Z}} \mid A^G(f_i, f_{i+1}) = 1, i \in \mathbb{Z}\}$$

defined by the matrix A^G. We denote by Λ_G^* the set of all admissible words of the subshift Λ_G (cf. [19]).

Lemma 3.1. *For $f_1, \dots, f_k \in E$, the following identity*

$$S_{f_k}^* S_{f_{k-1}}^* \cdots S_{f_2}^* S_{f_1}^* S_{f_1} S_{f_2} \cdots S_{f_{k-1}} S_{f_k} = A^G(f_1, f_2) A^G(f_2, f_3) \cdots A^G(f_{k-1}, f_k) S_{f_k}^* S_{f_k}$$

holds. If in particular $f_1 \cdots f_k \in \Lambda_G^*$, we have

$$S_{f_k}^* S_{f_{k-1}}^* \cdots S_{f_2}^* S_{f_1}^* S_{f_1} S_{f_2} \cdots S_{f_{k-1}} S_{f_k} = S_{f_k}^* S_{f_k}.$$

Proof. By using relations (3.1) recursively, the above identities are straightforward. □

Lemma 3.2. *For every X in B_n^G, there exists a vertex $v(X) \in V$ such that $\pi_G(X) = P_{v(X)}$.*

Proof. By Proposition 2.2 and Lemma 3.1, one sees that for every X in B_n^G, the element $\pi_G(X)$ is of the form

$$\pi_G(X) = S_{f_{k_1}}^* S_{f_{k_1}} S_{f_{k_2}}^* S_{f_{k_2}} \cdots S_{f_{k_m}}^* S_{f_{k_m}}$$

for some words $f_{k_1}, f_{k_2}, \cdots f_{k_m}$. Since $S_{f_{k_j}}^* S_{f_{k_j}} = P_{t(f_{k_j})}$, one has

$$\pi_G(X) = P_{t(f_{k_1})} P_{t(f_{k_2})} \cdots P_{t(f_{k_m})}.$$

As $\pi_G(X) \neq 0$ and $P_u P_v = 0$ for $u \neq v$, one obtains

$$t(f_{k_1}) = t(f_{k_2}) = \cdots = t(f_{k_m}).$$

By putting $v = t(f_{k_1})$, one concludes that $\pi_G(X) = P_v$. □

Put for $e \in E$

$$B_n^G[e] = \{X = (X_1, \dots, X_{2n}) \in B_n^G \mid (S_e^*, X_1, \dots, X_{2n}, S_e) \in B_{n+1}^G\}.$$

We note the following lemma

Lemma 3.3. *A word X in B_n^G belongs to $B_n^G[e]$ if and only if $v(X) = s(e)$.*

Proof. For $X = (X_1, \dots, X_{2n}) \in B_n^G$, it follows that

$$\pi_G((S_e^*, X_1, \dots, X_{2n}, S_e)) = S_e^* \pi_G(X) S_e = S_e^* P_{v(X)} S_e.$$

As $S_e^* P_{v(X)} S_e \neq 0$ if and only if $P_{v(X)} \geq S_e S_e^*$. The latter condition is equivalent to the condition $v(X) = s(e)$. Hence the word $(S_e^*, X_1, \dots, X_{2n}, S_e)$ belongs to B_{n+1}^G if and only if $v(X) = s(e)$. □

For $X = (X_1, \dots, X_{2n}) \in B_n^G$, the leftmost symbol X_1 must be S_e^* for some $e \in E$ and the rightmost symbol X_{2n} must be S_f for some $f \in E$. We denote e by $l(X)$ and f by $r(X)$ respectively. Hence X is of the form

$$X = (S_{l(X)}^*, X_2, \dots X_{2n-1}, S_{r(X)}).$$

As $S_{l(X)}^* S_{l(X)} = S_{r(X)}^* S_{r(X)} = P_{v(X)}$, one sees $t(l(X)) = t(r(X)) = v(X)$. The following property for brackets is well-known.

Lemma 3.4. *For a G-Catalan word $X = (X_1, \dots, X_{2n+2})$ in B_{n+1}^G, there uniquely exists $k \in \mathbb{N}$ with $0 \leq k \leq n$ such that*

$$(X_2, \dots, X_{2k+1}) \in B_k^G, \quad X_{2k+2} = X_1^*, \quad (X_{2k+3}, \dots, X_{2n+2}) \in B_{n-k}^G.$$

This lemma means by putting

$$Y = (X_2, \dots, X_{2k+1}), \qquad Z = (X_{2k+3}, \dots, X_{2n+2}),$$

the word X is decomposed as

$$X = (S_{l(X)}^*, Y, S_{l(X)}, Z), \qquad Y \in B_k^G, \qquad Z \in B_{n-k}^G,$$

in a unique way.

Lemma 3.5. *For $e, f \in E$, and $Y = (Y_1, \dots, Y_{2k}) \in B_k^G, Z = (Z_1, \dots, Z_{2(n-k)}) \in B_{n-k}^G$, the word $(S_f^*, Y_1, \dots, Y_{2k}, S_f, Z_1, \dots, Z_{2(n-k)}) \in W_{2n+2}^G$ belongs to $B_{n+1}^G[e]$ if and only if*

$$Y \in B_k^G[f], \quad Z \in B_{n-k}^G[e] \quad \text{and} \quad t(f) = s(e).$$

Proof. By Lemma 3.3, Y belongs to $B_k^G[f]$ if and only if $v(Y) = s(f)$, and Z belongs to $B_{n-k}^G[e]$ if and only if $v(Z) = s(e)$. As we have

$$\pi_G((S_f^*, Y_1, \dots, Y_{2k}, S_f, Z_1, \dots, Z_{2(n-k)})) = S_f^* P_{v(Y)} S_f P_{v(Z)},$$

the above element is not zero if and only if $v(Y) = s(f)$ and $v(Z) = t(f)$. The latter condition is equivalent to the conditions

$$Y \in B_k^G[f], \quad Z \in B_{n-k}^G[e] \quad \text{and} \quad t(f) = s(e).$$

□

Hence we have

Lemma 3.6. *For $e \in E$, the equality*

$$B^G_{n+1}[e] = \bigsqcup_{\substack{f \in E \\ A^G(f,e)=1}} \bigsqcup_{k=0}^{n} B^G_k[f] \times B^G_{n-k}[e]$$

holds through the correspondence

$$X = (S^*_f, Y, S_f, Z) \in B^G_{n+1}[e] \longrightarrow (Y, Z) \in B^G_k[f] \times B^G_{n-k}[e].$$

We set

$$c^G_0[e] = 1, \qquad c^G_n[e] = |B^G_n[e]| \quad \text{for} \quad n = 1, 2, \dots, \quad e \in E.$$

For $e, f \in E$, define an equivalence relation $e \sim f$ by the condition $s(e) = s(f)$. The equivalence relation $e \sim f$ implies $B^G_n[e] = B^G_n[f]$ and hence $c^G_n[e] = c^G_n[f]$. For a vertex $u \in V$, we define

$$B^G_n(u) = \{X \in B^G_n \mid v(X) = u\}$$

so that $B^G_n = \sqcup_{u \in V} B^G_n(u)$. We set for a vertex $u \in V$,

$$c^G_0(u) = 1, \qquad c^G_n(u) = |B^G_n(u)| \quad \text{for } n = 1, 2, \dots, \quad u \in V.$$

Therefore we have

Proposition 3.7.

(i) $c^G_n = \sum_{u \in V} c^G_n(u)$.

(ii) $c^G_{n+1}[e] = \sum_{k=0}^{n} c^G_{n-k}[e] \sum_{f \in E} A^G(f,e) c^G_k[f]$.

(iii) *if $s(e) = u$, we have $c^G_n(u) = c^G_n[e]$.*

Proof. The assertions are all obvious. □

4. Dyck paths associated with the graph G

We will define Dyck paths associated with a given directed graph G. We will enumerate them and define the sequence $d^G_n, n = 0, 1, \dots$ of numbers. We will prove that

$$d^G_n = c^G_n \qquad \text{for } n = 0, 1, \dots.$$

Let $G = (V, E)$ be a directed graph. Let G^* denote the transposed graph of G. The vertex set V^* of G^* is V and the edge set E^* of G^* is the edges reversing the directions of edges of G.

A Dyck path γ is a continuous broken line located in the upper half plane and consisting of vectors $(1,1)$ and $(1,-1)$ starting at the origin and ending at the x-axis (see Figure 2). For a Dyck path $\gamma = (\gamma_1, \dots, \gamma_{2n})$, where γ_i is one of vectors $(1,1)$ and $(1,-1)$, if γ_i is a vector $(1,1)$, there uniquely exists γ_{i+k} satisfying the following conditions:

(1) γ_{i+k} is a vector $(1,-1)$.

(2) $(\gamma_{i+1}, \gamma_{i+2}, \dots, \gamma_{i+k-1})$ is a Dyck path of length $k-1$ (hence $k-1$ is even).

We call the edge γ_{i+k} the partner of γ_i.

For an edge $e \in E$, we denote by e^* the edge of G^* obtained by reversing the direction of e. A *G-Dyck path* of length $2n$ is a Dyck path γ labeled $\{e^*, e \mid e \in E\}$ by the following rules:

(1) vectors $(1,1)$ are labeled e^* for $e \in E$,

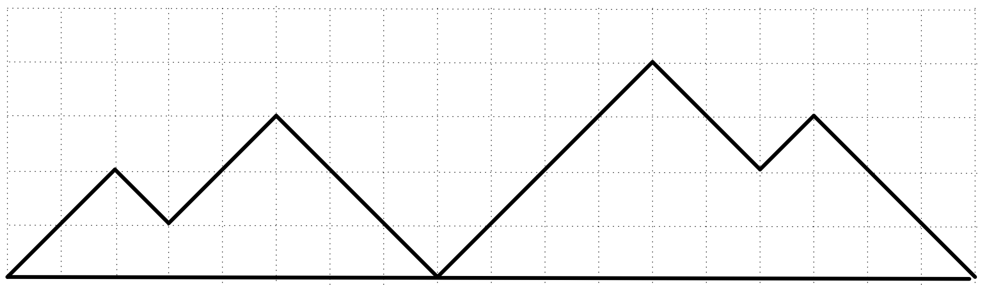

Figure 2

(2) vectors $(1,-1)$ are labeled e for $e \in E$,
(3) a vector $(1,1)$ labeled e^* follows a vector $(1,1)$ labeled f^* if and only if $t(f^*) = s(e^*)$ in G^*,
(4) a vector $(1,1)$ labeled e^* follows a vector $(1,-1)$ labeled f if and only if $t(f) = s(e^*)$ in G^*,
(5) a vector $(1,-1)$ labeled e follows a vector $(1,1)$ labeled f^* if and only if $e = f$,
(6) the partner of a vector $(1,1)$ labeled e^* is labeled by e.

Let D_n^G be the set of all G-Dyck paths of length $2n$. For $\gamma = (\gamma_1, \dots, \gamma_{2n}) \in D_n^G$, the leftmost vector γ_1 must be e^* for some $e^* \in E^*$, and the rightmost vector γ_{2n} must be f for some $f \in E$. We denote e^* and f by $l(\gamma)^*$ and $r(\gamma)$ respectively. We set

$$d_0^G = N, \qquad d_n^G = |D_n^G| \quad \text{for} \quad n = 1, 2, \dots.$$

The following lemmas are paralle to lemmas in the previous section.

Lemma 4.1. *For a G-Dyck path $\gamma = (\gamma_1, \dots, \gamma_{2n+2})$ in D_{n+1}^G, there uniquely exists $k \in \mathbb{N}$ with $0 \le k \le n$ such that*

$$(\gamma_2, \dots, \gamma_{2k+1}) \in D_k^G, \quad \gamma_{2k+2} = \gamma_1^*, \quad (\gamma_{2k+3}, \dots, \gamma_{2n+2}) \in D_{n-k}^G.$$

Put for $e \in E$

$$D_n^G[e] = \{(\gamma_1, \dots, \gamma_{2n}) \in D_n^G \mid (e^*, \gamma_1, \dots, \gamma_{2n}, e) \in D_{n+1}^G\}.$$

Lemma 4.2. *For $e, f \in E$, and $\eta = (\eta_1, \dots, \eta_{2k}) \in D_k^G, \zeta = (\zeta_1, \dots, \zeta_{2(n-k)}) \in D_{n-k}^G$, the path $(f^*, \eta_1, \dots, \eta_{2k}, f, \zeta_1, \dots, \zeta_{2(n-k)})$ belongs to $D_{n+1}^G[e]$ if and only if*

$$\eta \in D_k^G[f], \quad \zeta \in D_{n-k}^G[e] \quad \text{and} \quad t(f) = s(e).$$

We set

$$d_0^G[e] = 1, \qquad d_n^G[e] = |D_n^G[e]| \quad \text{for} \quad n = 1, 2, \dots, \quad e \in E.$$

Recall that the equivalence relation $\sim$ in E is defined by $e \sim f$ if $s(e) = s(f)$. Hence we have $e \sim f$ implies $D_n^G[e] = D_n^G[f]$ and hence $d_n^G[e] = d_n^G[f]$. For a vertex $u \in V$, we define

$$D_n^G(u) = \{\gamma \in D_n^G \mid t(r(\gamma)) = u\}$$

and

$$d_0^G(u) = 1, \qquad d_n^G(u) = |D_n^G[u]| \quad \text{for} \quad n = 1, 2, \dots, \quad u \in V.$$

Therefore we have

Proposition 4.3.

(i) $d_n^G = \sum_{u \in V} d_n^G(u)$,

(ii) $d_{n+1}^G[e] = \sum_{k=0}^{n} d_{n-k}^G[e] \sum_{f \in E} A^G(f,e) d_k^G[f]$,

(iii) *if* $s(e) = u$, *we have* $d_n^G(u) = d_n^G[e]$.

Proof. The assertions are all obvious. □

Therefore we have

Theorem 4.4. *For* $n = 0, 1, \ldots,$ *we have* $d_n^G[e] = c_n^G[e]$ *for* $e \in E$ *and* $d_n^G(u) = c_n^G(u)$ *for* $u \in V$*, so that*

$$d_n^G = c_n^G.$$

5. Trees associated with graph G

We will define trees associated with a given directed graph G. Let t_n^G be the numbers of such trees with n edges. We will prove that

$$t_n^G = c_n^G \qquad \text{for } n = 0, 1, \ldots.$$

A rooted tree is a plane tree with a distinguished vertex. The distinguished vertex is called the root (see Figure 3).

$n = 1$

$n = 2$

$n = 3$

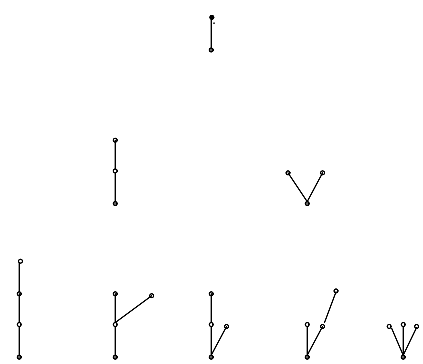

Figure 3

It is well-known that the ordinary Catalan numbers enumerate the number of the rooted trees. In this section we consider trees associated with graph G.

Let $G = (V, E)$ be a directed graph and $G^* = (V^*, E^*)$ the transposed graph of G. A *G-rooted tree* $\mathcal{T}$ with n edges is an oriented rooted tree with n edges satisfying the following conditions:

(1) each edge with vertices is labeled by edges with vertices of G^*,
(2) an edge e^* of $\mathcal{T}$ follows an edge f^* of $\mathcal{T}$ if and only if e^* follows f^* in the graph G^*.

Let T_n^G be the set of all G-rooted trees with n edges. We set

$$t_0^G = N, \qquad t_n^G = |T_n^G| \quad \text{for } n = 1, 2, \ldots.$$

For a vertex $u \in V$ and an edge $e \in E$, let $T_n^G(u)$ be the set of all G-rooted trees whose root is the vertex u, and $T_n^G[e]$ the set of all G-rooted trees whose root is the source of e. Put

$$t_0^G(u) = 1, \qquad t_n^G(u) = |T_n^G(u)| \quad \text{for } n = 1, 2, \dots,$$
$$t_0^G[e] = 1, \qquad t_n^G[e] = |T_n^G[e]| \quad \text{for } n = 1, 2, \dots.$$

Proposition 5.1. *For $n = 0, 1, \dots,$ we have $t_n^G[e] = d_n^G[e]$ for $e \in E$ and $t_n^G(u) = d_n^G(u)$ for $u \in V$, so that*

$$t_n^G = d_n^G.$$

Proof. For a G-Dyck path γ in D_n^G, by considering vectors $(1,1)$ in γ, one gets a G-rooted tree. This correspondence yields bijective mappings between D_n^G and T_n^G, between $D_n^G(u)$ and $T_n^G(u)$, and between $D_n^G[e]$ and $T_n^G[e]$. □

6. Generating functions

We will next study the generating functions of the sequance $c_n^G, n = 0, 1, \dots$. Let $f^G(x)$ be the generating function for the sequence $c_n^G, n = 0, 1, \dots,$ that is defined by

$$f^G(x) = \sum_{n=0}^{\infty} c_n^G x^n \tag{6.1}$$

as a formal power series. For a vertex $u, v \in V$, we denote by $A_G(v, u)$ the number of edges from v to u in G. By proposition 3.7 the following proposition holds.

Proposition 6.1. *For $n = 0, 1, \dots,$ we have*

$$c_{n+1}^G(u) = \sum_{k=0}^{n} c_{n-k}^G(u) \sum_{v \in V} A_G(v, u) c_k^G(v).$$

To study the sequence $c_n^G, n = 0, 1, \dots$ and its generating function $f^G(x)$, we provide the generating functions for the sequences $c_n^G(u), n = 0, 1, \dots$ for $u \in V$. Let $\{v_1, \dots, v_N\}$ be the vertex set V of G. We put

$$c_n^G(i) = c_n^G(v_i), \quad A_G(i, j) = A_G(v_i, v_j) \quad \text{for } i, j = 1, \dots, N.$$

Let $f_i^G(x)$ be the generating function for the sequence $c_n^G(i), n = 0, 1, \dots$. It is defined by

$$f_i^G(x) = \sum_{n=0}^{\infty} c_n^G(i) x^n, \qquad i = 1, \dots, N \tag{6.2}$$

as a formal power series. The preceding proposition implies the following equalities

$$c_{n+1}^G(i) = \sum_{k=0}^{n} c_{n-k}^G(i) \sum_{j=1}^{N} A_G(j, i) c_k^G(j), \qquad i = 1, \dots, N \tag{6.3}$$

so that we have

Proposition 6.2.

(i) $f^G(x) = \sum_{i=1}^{N} f_i^G(x)$,

(ii)

$$f_i^G(x) = 1 + x f_i^G(x) \sum_{j=1}^{N} A_G(j,i) f_j^G(x), \qquad \textit{for} \quad i = 1, \ldots, N. \tag{6.4}$$

Proof. (i) The equality is clear. (ii) By (6.3), one has

$$\begin{aligned} f_i^G(x) - 1 &= x \sum_{n=0}^{\infty} c_{n+1}^G(i) x^n \\ &= x \sum_{n=0}^{\infty} \sum_{k=0}^{n} c_{n-k}^G(i) x^{n-k} \sum_{j=1}^{N} A_G(j,i) c_k^G(j) x^k \\ &= x \sum_{j=1}^{N} \sum_{n=0}^{\infty} A_G(j,i) \sum_{k=0}^{n} c_{n-k}^G(i) x^{n-k} c_k^G(j) x^k \\ &= x \sum_{j=1}^{N} f_i^G(x) A_G(j,i) f_j^G(x). \end{aligned}$$

□

Lemma 6.3.

$$c_n^G(i) \le \|A_G\|_1^n c_n, \qquad i = 1, \ldots, N, \quad n = 0, 1, \ldots$$

where $\|A_G\|_1 = \max_{1 \le i \le N} \sum_{j=1}^{N} A_G(j,i)$ *and* $c_n = \frac{1}{n+1}\binom{2n}{n}$ *the Catalan number.*

Proof. We will prove the above identity by induction. Fix $i = 1, \ldots, N$. For $n = 0$, the inequality is trivial. For $n = 1$, one sees that $c_1^G(i) = \sum_{j=1}^{N} A_G(j,i) \le \|A_G\|_1$. Assume that the inequality holds for all $n \le k$. As $\sum_{l=0}^{k} c_{k-l} c_l = c_{k+1}$, it follows that

$$\begin{aligned} c_{k+1}^G(i) &= \sum_{l=0}^{k} c_{k-l}^G(i) \sum_{j=1}^{N} A_G(j,i) c_l^G(j) \\ &\le \sum_{l=0}^{k} \|A_G\|_1^{k-l} c_{k-l} \sum_{j=1}^{N} A_G(j,i) \|A_G\|_1^l c_l \\ &= \|A_G\|_1^{k+1} c_{k+1} \end{aligned}$$

Hence the desired inequality holds. □

We denote by R_i^G the radius $\frac{1}{\limsup_{n\to\infty} \sqrt[n]{c_n^G(i)}}$ of convergence of $f_i^G(x)$.

Lemma 6.4. *Suppose that G is irreducible.*

(i) $R_i^G = R_j^G, \quad i, j = 1, \ldots, N.$

(ii) $\frac{1}{4\|A_G\|_1} \le R_i^G, \quad i = 1, \ldots N.$

Proof. Put $\alpha_i = \limsup_{n\to\infty} \sqrt[n]{c_n^G(i)}$.

(i) By the relation (6.3), one has $c_{n+1}^G(i) \ge c_0^G(i) A_G(j,i) c_n^G(j)$. Assume that $A_G(j,i) \ne 0$. As $c_0^G(i) = 1$, one has $c_{n+1}^G(i) \ge c_n^G(j)$ so that $\alpha_i \ge \alpha_j$. Since G is irreducible, one sees that $\alpha_i = \alpha_j$ for all $i, j = 1, \ldots, N$.

(ii) It is well-known that $\limsup_{n\to\infty} \sqrt[n]{c_n} = 4$. By the preceding lemma, the inequality $\alpha_i \le 4\|A_G\|_1$ is immediate. □

We note that the value $\frac{1}{4\|A_G\|_1}$ is not best possible in general (see Section 8).

Therefore the functions $f_i^G(x)$ defined by (6.2) exists in a neighborhood of the origin, and they satisfy the relations (6.4). Conversely, the following proposition states that the functions are uniquely determined by only the relations (6.4).

Theorem 6.5. *A family $f_i(x), i = 1, \dots, N$ of functions satisfying the relations*

$$f_i(x) = 1 + xf_i(x) \sum_{j=1}^{N} A_G(j,i) f_j(x), \qquad i = 1, \dots, N \tag{6.5}$$

uniquely exists in a neighborhood of the origin, and they are differentiable.

Proof. We first note that by the above equalities one has

$$f_i(0) = 1, \qquad i = 1, \dots, N.$$

Consider the family of polynomials defined by

$$F_i(x, y_1, \dots, y_N) = xy_i \sum_{j=1}^{N} A_G(j,i) y_j - y_i + 1, \qquad i = 1, \dots, N.$$

Put $\mathbf{y} = (y_1, \dots, y_N) \in \mathbb{R}^N$. We set the $\mathbb{R}^N$-valued C^∞-function $\mathbb{F}$ on $\mathbb{R} \times \mathbb{R}^N$

$$\mathbb{F}(x, \mathbf{y}) = \begin{bmatrix} F_1(x, y_1, \dots, y_N) \\ F_2(x, y_1, \dots, y_N) \\ \vdots \\ F_N(x, y_1, \dots, y_N) \end{bmatrix}.$$

We note that $\mathbb{F}(0, 1, \dots, 1) = \mathbf{0}$. Since

$$\frac{\partial F_i}{\partial y_j} = \begin{cases} xy_i A(i,i) + x \sum_{k=1}^{N} A(k,i) y_k - 1 & \text{if } j = i, \\ xA(j,i) y_i & \text{if } j \neq i, \end{cases}$$

one has the Jacobian matrix of $\mathbb{F}$ as

$$\begin{aligned} \frac{\partial \mathbb{F}}{\partial \mathbf{y}} = & \begin{bmatrix} \frac{\partial F_1}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial y_N} \\ \vdots & & \vdots \\ \frac{\partial F_N}{\partial y_1} & \cdots & \frac{\partial F_N}{\partial y_N} \end{bmatrix} \\ = & x \begin{bmatrix} y_1 & & \\ & \ddots & \\ & & y_N \end{bmatrix} \begin{bmatrix} {}^tA(1,1) & \cdots & {}^tA(1,N) \\ \vdots & & \vdots \\ {}^tA(N,1) & \cdots & {}^tA(N,N) \end{bmatrix} \\ & + x \begin{bmatrix} \sum_{k=1}^{N} {}^tA(1,k) y_k & & \\ & \ddots & \\ & & \sum_{k=1}^{N} {}^tA(N,k) y_k \end{bmatrix} - \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}. \end{aligned}$$

Hence we have

$$\frac{\partial \mathbb{F}}{\partial \mathbf{y}}(0, 1, \dots, 1) = - \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} \neq 0.$$

By the implicit function theorem, one sees the assertion. $\square$

We note that the functions $f_1(x), \dots, f_N(x)$ are holomorphic in a neighborhood of the origin.

Let $f_1(x), \dots, f_N(x)$ be a family of functions satisfying the equalities (6.5). They are uniquely defined by a neighborhood of the origin by Proposition 6.5. Let $f_i^{(n)}(x)$ be the n-th derivative of f_i. Since the family of the functions is unique, the i-th G-Catalan numbers $c_n^G(i)$ are given by

$$c_n^G(i) = \frac{f_i^{(n)}(0)}{n!}, \qquad n = 0, 1, \dots.$$

We henceforth assume that G is irreducible. We denote by R_G the radius R_i^G of convergence of $f_i^G(x)$ as in Lemma 6.4. We put $I_{R_G} = \{x \in \mathbb{R} \mid |x| < R_G\}$.

Lemma 6.6.

(i) *$f_i(x) \neq 0$ for $x \in I_{R_G}$ and $f_i(x) > 1$ for $0 < x \in I_{R_G}$.*

(ii) *There exists $M > 0$ such that $|f_i(x)| < M$ for all $x \in I_{R_G}$, $i = 1, \dots, N$.*

Proof. (i) Since $f_i(x) = f_i^G(x)$ on $x \in I_{R_G}$, the assertion (i) is clear by definition of $f_i^G(x)$.

(ii) Suppose that there exists $j = 1, \dots, N$ and $x_n \in I_{R_G}$ such that $\lim_{n\to\infty} |f_j(x_n)| = \infty$. Since $|f_j(x_n)| \le f_j(|x_n|)$, we may assume that $x_n > 0$ by considering $|x_n|$ instead of x_n. Take $i = 1, \dots, N$ such that $A_G(j, i) = 1$. As we have

$$f_i(x_n) - 1 \ge x_n f_i(x_n) f_j(x_n),$$

the inequality

$$1 \ge 1 - \frac{1}{f_i(x_n)} \ge x_n f_j(x_n) \ge 0$$

holds by (i). By hypothesis, one sees that $\lim_{n\to\infty} x_n = 0$, a contradiction to the fact $1 = f_j(0)$ with the continuity of f_j at 0. □

Proposition 6.7. *The functions $f_i(x), i = 1, \dots, N$ satisfying* (6.5) *can be defined at $x = R_G$, and they are lower semi-continuous at $x = R_G$.*

Proof. Take an increasing sequence x_n in I_{R_G} such that $x_n \uparrow R_G$. For each $i = 1, \dots, N$, the sequence $\{f_i(x_n)\}_{n=0,1,\dots}$ is increasing and bounded by the preceding lemma so that the functions $f_i(x)$ can be defined at $x = R_G$, and they are lower-continuous at $x = R_G$. □

Therefore we have

Theorem 6.8. *The family $f_i(x), i = 1, \dots, N$ of functions satisfying* (6.5) *uniquely exists on $[-R_G, R_G]$ for some $R_G > \frac{1}{4\|A\|_1}$. They can not be defined outside of the interval $[-R_G, R_G]$.*

We note the following proposition.

Proposition 6.9. *If $\{j \mid A(j, i_1) = 1\} = \{j \mid A(j, i_2) = 1\}$, then we have $f_{i_1}(x) = f_{i_2}(x)$.*

Proof. Put $g_i(x) = \sum_{j=1}^N A_G(j, i) f_j(x)$. As $f_i(x)$ is equal to $\frac{1}{1 - x g_i(x)}$ for $x \neq 0$, one sees $f_{i_1}(x) = f_{i_2}(x)$ for $x \neq 0$ by hypothesis. As $f_{i_1}(0) = f_{i_2}(0) = 1$, we have $f_{i_1}(x) = f_{i_2}(x)$. □

7. The radius of convergence of the generating functions

In this section, we will study how to find the radius R_G of convergence of the functions $f_i(x), i = 1, \ldots, N$ satisfying (6.5). For an $N \times N$ matrix A and $t = (t_i)_{i=1}^N \in \mathbb{R}^N$, we set

$$x(t)_i = t_i - \sum_{k=1}^{N} t_k A(k,i) t_i, \qquad (tAt)(i,j) = t_i A(i,j) t_j.$$

for $i, j = 1, \ldots, N$. Hence we have an $N \times N$ matrix tAt. We set

$$C_A = \{t = (t_i)_{i=1}^N \in \mathbb{R}^N \mid t_i > 0,\, x(t)_i = x(t)_j > 0 \text{ for } i, j = 1, \ldots, N\}.$$

We note that

$$C_{A_G} = \{(x f_i^G(x))_{i=1}^N \in \mathbb{R}^N \mid 0 < x \in I_{R_G}\}$$

by the relations (6.4) and Theorem 6.8. Let $\widetilde{A}$ be the $N \times N$ matrix defined by

$$\widetilde{A}(i,j) = L_A - A(i,j), \qquad i, j = 1, \ldots, N$$

where $L_A = \max_{i,j} A(i,j)$. For $t = (t_i)_{i=1}^N \in C_A$, the value $x(t)_i$ does not depend on $i = 1, \ldots, n$. We denote it by $x(t)$. We say that A satisfies *condition* (C) if

$$\ker(tAt - x(t)) \cap \ker(t\widetilde{A}t + x(t)) = \{0\}$$

for all $t = (t_i)_{i=1}^N \in C_A$.

Lemma 7.1. *A matrix A satisfies condition (C) if and only if*

$$\ker(tAt - x(t)) \cap \mathbf{1}^\perp = \{0\}.$$

for all $t = (t_i)_{i=1}^N \in C_A$, where $\mathbf{1}^\perp = \{(r_i)_{i=1}^N \in \mathbb{R}^N \mid \sum_{i=1}^N r_i = 0\}$.

Proof. We will show that for $(r_i)_{i=1}^N \in \ker(tAt - x(t))$, the vector $(r_i)_{i=1}^N$ belongs to $\ker(t\widetilde{A}t + x(t))$ if and only if $\sum_{i=1}^N r_i = 0$. As $(r_i)_{i=1}^N \in \ker(tAt - x(t))$, one has

$$\sum_{i=1}^N x(t) r_i = \sum_{i=1}^N t_i r_i - \sum_{i,k=1}^N t_k A(k,i) t_i r_i = \sum_{i=1}^N t_i r_i - \sum_{i=1}^N x(t) r_i$$

so that

$$2x(t) \sum_{i=1}^N r_i = \sum_{i=1}^N t_i r_i.$$

Since $x(t) > 0$, one knows that $\sum_{i=1}^N r_i = 0$ if and only if $\sum_{i=1}^N t_i r_i = 0$. Now

$$t\widetilde{A}t \begin{bmatrix} r_1 \\ \vdots \\ r_N \end{bmatrix} = \begin{bmatrix} L_A t_1 \sum_{j=1}^N t_j r_j - t_1 \sum_{j=1}^N A(1,j) t_j r_j \\ \vdots \\ L_A t_N \sum_{j=1}^N t_j r_j - t_N \sum_{j=1}^N A(N,j) t_j r_j \end{bmatrix} = L_A \sum_{j=1}^N t_j r_j \begin{bmatrix} t_1 \\ \vdots \\ t_N \end{bmatrix} - x(t) \begin{bmatrix} r_1 \\ \vdots \\ r_N \end{bmatrix}$$

Hence $\sum_{i=1}^N r_i = 0$ if and only if $(r_i)_{i=1}^N$ belongs to $\ker(t\widetilde{A}t + x(t))$. $\square$

It is easy to see that the matrices

$$[N], \quad \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

satisfy condition (C).

We will prove the following theorem:

Theorem 7.2. *Suppose that a matrix A_G satisfies condition (C). Let $f_i(x), i = 1, \dots, N$ be the functions satisfying (6.5). If a real number $x_0 \in \mathbb{R}$ is the radius R_G of convergence of the functions $f_i(x)$, then there exist positive real numbers $t_1, \dots, t_N$ such that*

$$\det(tA_Gt - x_0) = 0, \qquad x_0 = t_i - \sum_{j=1}^{N} t_j A_G(j,i) t_i \quad \text{for } i = 1, \dots, N$$

where tA_Gt is the $N \times N$ matrix defined by $tA_Gt = [t_i A_G(i,j) t_j]_{i,j=1,\dots N}$. In particular, there exists an eigenvector $[s_i]_{i=1,\dots,N}$ of the matrix tA_Gt for the eigenvalue x_0 with $\sum_{i=1}^{N} s_i = 1$ such that

$$x_0 = \frac{1}{2} \sum_{i=1}^{N} t_i s_i.$$

Therefore the radius of convergence of $f_i(x)$ is algebraically determined as a solution of an eigenvalue problem for the matrix tA_Gt with some conditions.

Proof. Put $t_i = x f_i(x), i = 1, \dots, N$ in (6.5) so that we have equalities

$$x = t_i - t_i \sum_{j=1}^{N} A_G(j,i) t_j, \qquad i = 1, \dots, N. \tag{7.1}$$

This implies that $(t_i)_{i=1}^N$ belongs to C_A for $0 < x \in I_{R_G}$. Consider $x = x(t_1, \dots, t_N)$ as a function of $(t_1, \dots, t_N)$ so that the radius R_G of convergence is the maximum value of the function $x = x(t_1, \dots, t_N)$. Put

$$\psi_i(t_1, \dots, t_N) = t_i - t_i \sum_{j=1}^{N} A_G(j,i) t_j, \qquad i = 1, \dots, N$$

and

$$\begin{aligned} f(t_1, \dots, t_N) &= \psi_1(t_1, \dots, t_N), \\ g_2(t_1, \dots, t_N) &= \psi_1(t_1, \dots, t_N) - \psi_2(t_1, \dots, t_N), \\ &\cdots \\ g_N(t_1, \dots, t_N) &= \psi_1(t_1, \dots, t_N) - \psi_N(t_1, \dots, t_N). \end{aligned}$$

Hence the radius R_G is obtained by solving the constrained extremal problem of f with constrained conditions:

$$g_2(t_1, \dots, t_N) = \cdots = g_N(t_1, \dots, t_N) = 0.$$

Suppose that

$$\sum_{j=2}^{N} c_j \frac{\partial g_j}{\partial t_i}(t_1, \dots, t_N) = 0, \qquad i = 1, \dots, N$$

for some $c_j \in \mathbb{R}, j = 2, \dots, N$. One then has

$$\begin{bmatrix} \frac{\partial\psi_1}{\partial t_1}(t_1,\dots,t_N) & \frac{\partial\psi_2}{\partial t_1}(t_1,\dots,t_N) & \cdots & \frac{\partial\psi_N}{\partial t_1}(t_1,\dots,t_N) \\ \frac{\partial\psi_1}{\partial t_2}(t_1,\dots,t_N) & \frac{\partial\psi_2}{\partial t_2}(t_1,\dots,t_N) & \cdots & \frac{\partial\psi_N}{\partial t_2}(t_1,\dots,t_N) \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial\psi_1}{\partial t_N}(t_1,\dots,t_N) & \frac{\partial\psi_2}{\partial t_N}(t_1,\dots,t_N) & \cdots & \frac{\partial\psi_N}{\partial t_N}(t_1,\dots,t_N) \end{bmatrix} \begin{bmatrix} \sum_{j=2}^N c_j \\ -c_2 \\ \vdots \\ -c_N \end{bmatrix} = 0.$$

As one sees

$$\frac{\partial\psi_i}{\partial t_j} = \begin{cases} 1 - \sum_{k=1}^N A_G(k,i)t_k - A_G(i,i)t_i & \text{if } i = j \\ -A_G(j,i)t_i & \text{if } i \neq j, \end{cases}$$

by putting

$$r_1 = \sum_{j=2}^N c_j, \quad r_2 = -c_2, \quad \dots, \quad r_N = -c_N,$$

one has

$$(1 - \sum_{k=1}^N t_k A_G(k,i))r_i = \sum_{j=1}^N A_G(i,j)t_j r_j, \qquad i = 1, \dots, N.$$

By (7.1), one has

$$x \begin{bmatrix} r_1 \\ \vdots \\ r_N \end{bmatrix} = tA_G t \begin{bmatrix} r_1 \\ \vdots \\ r_N \end{bmatrix}, \qquad \sum_{i=1}^N r_i = 0.$$

Since the matrix A_G satisfies condition (C), one gets $r_i = 0, i = 1, \dots, N$. This means that the rank of the matrix $[\frac{\partial g_j}{\partial t_i}(t_1,\dots,t_N)]_{i=1,\dots,N,\, j=2,\dots,N}$ is $N-1$. Now suppose that the function $f(x)$ takes its maximum value x_0 at $(t_1,\dots,t_N)$ under the conditions that $g_i(t_1,\dots,t_N) = 0$ for $i = 2,\dots,N$. Then there exists real numbers $\lambda_1,\dots,\lambda_N$ such that

$$\frac{\partial f}{\partial t_i}(t_1,\dots,t_N) + \sum_{j=2}^N \lambda_j \frac{\partial g_j}{\partial t_i}(t_1,\dots,t_N) = 0, \qquad i = 1,\dots,N.$$

It then follows that

$$\frac{\partial\psi_1}{\partial t_i}(t_1,\dots,t_N) + \sum_{j=2}^N \lambda_j (\frac{\partial\psi_1}{\partial t_i}(t_1,\dots,t_N) - \frac{\partial\psi_j}{\partial t_i}(t_1,\dots,t_N)) = 0$$

so that

$$\begin{bmatrix} \frac{\partial\psi_1}{\partial t_1}(t_1,\dots,t_N) & \frac{\partial\psi_2}{\partial t_1}(t_1,\dots,t_N) & \cdots & \frac{\partial\psi_N}{\partial t_1}(t_1,\dots,t_N) \\ \frac{\partial\psi_1}{\partial t_2}(t_1,\dots,t_N) & \frac{\partial\psi_2}{\partial t_2}(t_1,\dots,t_N) & \cdots & \frac{\partial\psi_N}{\partial t_2}(t_1,\dots,t_N) \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial\psi_1}{\partial t_N}(t_1,\dots,t_N) & \frac{\partial\psi_2}{\partial t_N}(t_1,\dots,t_N) & \cdots & \frac{\partial\psi_N}{\partial t_N}(t_1,\dots,t_N) \end{bmatrix} \begin{bmatrix} 1 + \sum_{j=2}^N \lambda_j \\ -\lambda_2 \\ \vdots \\ -\lambda_N \end{bmatrix} = 0.$$

Put

$$s_1 = 1 + \sum_{j=2}^N \lambda_j, \quad s_2 = -\lambda_2, \quad \dots, \quad s_N = -\lambda_N.$$

Similarly to the above discussions, we have

$$x_0 \begin{bmatrix} s_1 \\ \vdots \\ s_N \end{bmatrix} = tA_G t \begin{bmatrix} s_1 \\ \vdots \\ s_N \end{bmatrix}, \qquad \sum_{j=1}^{N} s_j = 1.$$

In this case we have

$$x_0 = \sum_{i=1}^{N} t_i s_i - \sum_{i,j=1}^{N} t_j A_G(j,i) t_i s_i = \sum_{i=1}^{N} t_i s_i - x_0.$$

so that we get

$$x_0 = \frac{1}{2} \sum_{i=1}^{N} t_i s_i.$$

Therefore we obtain the assertion. □

8. Integral formulae of the G-Catalan numbers

The G-Catalan numbers $c_n^G(i)$ are coefficients of its generating functions $f_i^G(x)$ so that one has

$$c_n^G(i) = \frac{1}{n!} \frac{d^n f_i^G(x)}{dx^n}(0) = \frac{1}{2\pi\sqrt{-1}} \int_C \frac{f_i^G(z)}{z^{n+1}} dz$$

where C denotes a positively oriented closed curve around the origin in the complex plane. In this section, we present a formula of $c_n^G(i)$ by using the relations (6.4). We set

$$F_i(w_1, \dots, w_N) = (w_i + 1) \sum_{j=1}^{N} A_G(j,i)(w_j + 1),$$
$$F_i^n(w_1, \dots, w_N) = F_i(w_1, \dots, w_N)^n, \qquad i = 1, \dots, N.$$

We will prove the following integral formulae

Theorem 8.1.

$$c_n^G(i) = \frac{1}{2\pi\sqrt{-1}n} \int_C \frac{F_j^n(w_1, \dots, w_N)}{w_j^n} dw_i, \qquad i,j = 1, \dots, N, \tag{8.1}$$

where the above integral is a contour integral along a positively oriented closed curve C around the origin in the complex plane.

Proof. Put $w_i(x) = f_i^G(x) - 1$. The relations (6.4) go to

$$w_i(x) = x \sum_{j=1}^{N} (w_j(x) + 1) A_G(j,i)(w_i(x) + 1) = x F_i(w_1(x), \dots, w_N(x)).$$

As

$$\lim_{x \to 0} \frac{w_i(x)}{x} = F_i(w_1(0), \dots, w_N(0)) = \sum_{j=1}^{N} A_G(j,i) > 0,$$

the rotation number of $w_i(x)$ is 1. One sees that

$$w_i'(z) = f_i'(z)$$

and

$$\frac{1}{z} = \frac{F_j(w_1(z), \dots, w_N(z))}{w_j(z)}.$$

It then folows that

$$\begin{aligned} c_n^G(i) &= \frac{1}{2\pi\sqrt{-1}n} \int_C \frac{f_i'(z)}{z^n} dz \\ &= \frac{1}{2\pi\sqrt{-1}n} \int_C \frac{F_j^n(w_1(z), \dots, w_N(z))}{w_j^n(z)} w_i'(z) dz. \end{aligned}$$

□

9. Catalan numbers associated to KMS states

In this section, we enumerate G-Catalan numbers by using KMS states on the Cuntz-Krieger algebra $\mathcal{O}_{A^G}$ for the gauge actions. We denote by $\mathbb{T}$ the group of complex numbers with modulus one. The gauge action α^G is an action of $\mathbb{T}$ to the automorphisms on $\mathcal{O}_{A^G}$ defined by $\alpha_z^G(S_e) = zS_e$ for $z \in \mathbb{T}, e \in E$. For a real number $\beta \in \mathbb{R}$, a state φ on $\mathcal{O}_{A^G}$ is called a KMS state at inverse temperature β if the following equality holds

$$\varphi(ab) = \varphi(b\alpha_{t+i\beta}^G(a)), \qquad t \in \mathbb{R}$$

for a in the dense analytic elements of $\mathcal{O}_{A^G}$ and $b \in \mathcal{O}_{A^G}$. In [7], it has been proved that under the condition that the matrix A^G is aperiodic, KMS state exists if and only if β is $\log r_G$, and the admitted KMS state is unique, where r_G is the Perron-Frobenius eigenvalue for the matrix A^G. In what follows, we fix a KMS state φ on $\mathcal{O}_{A^G}$. We define the (G, φ)-Catalan numbers $c_n^{G,\varphi}, n = 0, 1, \dots$ by setting

$$c_0^{G,\varphi} = 1, \qquad c_n^{G,\varphi} = \sum_{X \in B_n^G} \varphi(\pi_G(X)), \quad n = 1, 2, \dots.$$

These numbers are not necessarily integers. For $i = 1, \dots, N$, we put the (G, φ)-Catalan numbers $c_n^{G,\varphi}(i), n = 0, 1, \dots$ rooted at the vertex v_i by setting

$$c_0^{G,\varphi}(i) = \varphi(P_{v_i}), \qquad c_n^{G,\varphi}(i) = \sum_{X \in B_n^G(v_i)} \varphi(\pi_G(X)), \quad n = 1, 2, \dots.$$

By the preceding discussions, one has

Lemma 9.1. *For $n = 0, 1, \dots,$ and $i = 1, \dots, N$, we have*

(i) $c_n^{G,\varphi} = \sum_{i=1}^N c_n^{G,\varphi}(i)$.

(ii) $c_n^{G,\varphi}(i) = c_n^G(i)\varphi(P_{v_i})$.

(iii) $c_{n+1}^{G,\varphi}(i) = \sum_{k=0}^n \sum_{j=1}^N c_k^G(j) A_G(j, i) c_{n-k}^{G,\varphi}(i)$.

Proof. (i) is clear.

(ii) For $X \in B_n^G$, the word X belongs to $B_n^G(i)$ if and only if $v(X) = v_i$. The latter condition is equivalent to the condition $\pi_G(X) = P_{v_i}$. This implies the assertion (ii).

(iii) The desired equality comes from (ii) with (6.3). □

The generating functions $f^{G,\varphi}(x)$ and $f_i^{G,\varphi}(x)$ for the sequences $c_n^{G,\varphi}$ and $c_n^{G,\varphi}(i)$ respectively are also defined by

$$f^{G,\varphi}(x) = \sum_{n=0}^{\infty} c_n^{G,\varphi} x^n, \qquad f_i^{G,\varphi}(x) = \sum_{n=0}^{\infty} c_n^{G,\varphi}(i) x^n \qquad \text{for } i = 1, \dots, N.$$

Then the following lemma is direct from the preceding lemma.

Lemma 9.2. *For $i = 1, \dots, N$, we have*

(i) $f^{G,\varphi}(x) = \sum_{j=1}^{N} f_j^{G,\varphi}(x)$.

(ii) $f_i^{G,\varphi}(x) = f_i^{G}(x)\varphi(P_{v_i})$.

(iii) $f_i^{G,\varphi}(x) = \varphi(P_{v_i}) + x \sum_{j=1}^{N} f_j^{G}(x) A_G(j,i) f_i^{G,\varphi}(x)$.

The radii of convergence of the functions $f^{G,\varphi}(x), f_i^{G,\varphi}(x)$ coincide with that of $f^G(x)$ if G is irreducible.

Therefore we have

Proposition 9.3. *Suppose that A^G is aperiodic. Let $[t_e]_{e\in E}$ be the positive eigenvector for the Perron-Frobenius eigenvalue r_G of the matrix $[A^G(e,f)]_{e,f\in E}$ satisfying $\sum_{e\in E} t_e = 1$. For a vertex $v_i \in V, i = 1, \dots, N$, take an edge $e_i \in E$ such that $t(e_i) = v_i$. Then we have*

$$c_n^{G,\varphi}(i) = r_G c_n^{G}(i) t_{e_i}, \qquad c_n^{G,\varphi} = r_G \sum_{i=1}^{N} c_n^{G}(i) t_{e_i}$$

and hence

$$f_i^{G,\varphi}(x) = r_G f_i^{G}(x) t_{e_i}, \qquad f^{G,\varphi}(x) = r_G \sum_{i=1}^{N} f_i^{G}(x) t_{e_i}.$$

Proof. As in [7], the vector $[\varphi(S_e S_e^*)]_{e\in E}$ is the unique positive eigenvector for the Perron-Frobenius eigenvalue r_G of the matrix $[A^G(e,f)]_{e,f\in E}$ satisfying $\sum_{f\in E} \varphi(S_f S_f^*) = 1$. Hence we have $\varphi(S_e S_e^*) = t_e$ for $e \in E$. It follows that

$$\varphi(S_e^* S_e) = \sum_{f\in E} A^G(e,f)\varphi(S_f S_f^*) = r_G t_e$$

and $\varphi(P_{v_i}) = \varphi(S_{e_i}^* S_{e_i}) = r_G t_{e_i}$. By the preceding lemma, one has $c_n^{G,\varphi}(i) = r_G c_n^{G}(i) t_{e_i}$ and $c_n^{G,\varphi} = r_G \sum_{i=1}^{N} c_n^{G}(i) t_{e_i}$. □

10. Examples

1. Consider the following graph G_1 having N-loops with a single vertex.

Since $A_{G_1} = [N]$, we have

$$f^{G_1}(x) = f_1^{G_1}(x) \quad \text{and} \quad f_1^{G_1}(x) - 1 = xN f_1^{G_1}(x)^2$$

so that

$$f^{G_1}(x) = \frac{1 - \sqrt{1 - 4Nx}}{2Nx}.$$

We choose the minus sign before the root of $1 - 4Nx$ because $\lim_{x\to 0} f^{G_1}(x) = 1$. Although one directly konows that the radius R_{G_1} of convergence of $f^{G_1}(x)$ is equal

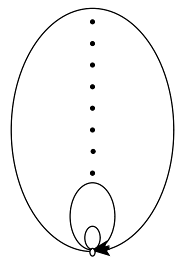

FIGURE 4

to $\frac{1}{4N}$ because of $1-4xN \geq 0$, we will see $R_{G_1} = \frac{1}{4N}$ by using Theorem 7.2 as in the following way. Suppose that a positive real number x_0 is R_{G_1}. There exists $t>0$ such that

$$tNt - x_0 = 0, \qquad x_0 = t - tNt.$$

By these equations we have $t = \frac{1}{2N}$ and $x_0 = \frac{1}{4N}$. We will next compute $c_n^{G_1}$. Althogh by the Newton Binomial formula for $(1-4Nx)^{\frac{1}{2}}$ one may easily compute $c_n^{G_1}$, we will use Theorem 8.1 as follows. Put $F(w) = N(w+1)^2$ so that we have

$$c_n^{G_1} = \frac{1}{2\pi n\sqrt{-1}} \int_C \frac{F^n(w)}{w^n} dw = \frac{1}{n} N^n \binom{2n}{n-1} = N^n c_n.$$

Since $\varphi(P_{v_1}) = 1$, the equality $c_n^{G_1,\varphi} = c_n^{G_1}$ holds.

2. Let G_2 be the directed graph with N vertices such that for any ordered pair of two vertices u, v, there uniquely exists an edge from u to v. Hence the transition matrix A_{G_2} is $\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix}$. The generating functions $f_i^{G_2}$ satisfy the following equalities

$$f_i^{G_2}(x) - 1 = x f_i^{G_2}(x) \sum_{j=1}^{N} f_j^{G_2}(x), \qquad i = 1, \ldots, N.$$

By Proposition 6.9, one has $f_i^{G_2}(x) = f_j^{G_2}(x)$ for $i, j = 1, \ldots, N$. By the above equalities, we have

$$f_i^{G_2}(x) - 1 = xN {f_i^{G_2}}^2(x).$$

Therefore we have

$$f_i^{G_2}(x) = f^{G_1}(x) = \frac{1 - \sqrt{1-4Nx}}{2Nx}.$$

Hence we have

$$R_{G_2} = R_{G_1} = \frac{1}{4N},$$

$$f^{G_2}(x) = N f_i^{G_2}(x) = \frac{1-\sqrt{1-4Nx}}{2x},$$

$$c_n^{G_2} = \sum_{i=1}^{N} c_n^{G_2}(i) = N c_n^{G_1} = \frac{1}{n} N^{n+1} \binom{2n}{n-1} = N^{n+1} c_n$$

By the equality $\varphi(P_{v_i}) = \frac{1}{N}$, we note

$$c_n^{(G_2,\varphi)} = c_n^{G_2} \cdot \frac{1}{N} = c_n^{G_1} = N^n c_n.$$

3. Let G_3 be the directed graph with $A_{G_3} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

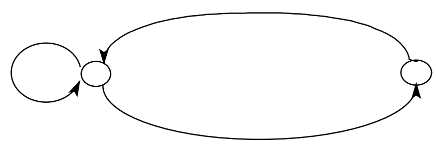

FIGURE 5

We then have

$$f^{G_3}(x) = f_1^{G_3}(x) + f_2^{G_3}(x).$$

Put $f_i(x) = f_i^{G_3}(x), i = 1, 2$. They satisfy the following relations:

$$\begin{aligned} f_1(x) - 1 &= x(f_1(x) + f_2(x))f_1(x), \\ f_2(x) - 1 &= xf_1(x)f_2(x) \end{aligned}$$

so that the equalities

$$f_2(x)^2 = f_1(x), \qquad xf_2(x)^3 - f_2(x) + 1 = 0 \tag{10.1}$$

hold. We will compute the radius R_{G_3} of convergence of the function $f^{G_3}(x)$. Suppose that a positive real number x_0 is R_{G_3}. By Theorem 7.2, there exist $t_1, t_2 > 0$ such that

$$\det\left(\begin{bmatrix} t_1^2 & t_1 t_2 \\ t_2 t_1 & 0 \end{bmatrix} - \begin{bmatrix} x_0 & 0 \\ 0 & x_0 \end{bmatrix}\right) = 0,$$
$$x_0 = t_1 - (t_1 + t_2)t_1, \qquad x_0 = t_2 - t_1 t_2.$$

One easily sees that $t_1 = \frac{1}{3}, \quad t_2 = \frac{2}{9}$ and hence $x_0 = \frac{4}{27}$. We will next compute $c_n^{G_3}$ by using Theorem 8.1. Put $w_i(x) = f_i(x) - 1, i = 1, 2$ and

$$F_1(w_1, w_2) = (w_1 + 1)\{(w_1 + 1) + (w_2 + 1)\}, \qquad F_2(w_1, w_2) = (w_1 + 1)(w_2 + 1).$$

We then have

$$w_i(x) = xF_i(w_1(x), w_2(x)), \qquad i = 1, 2.$$

It follows that by (10.1)

$$\begin{aligned} F_2^n(w_1(z), w_2(z)) &= (w_2(z) + 1)^{3n}, \\ w_1'(z) &= 2(w_2(z) + 1)w_2'(z). \end{aligned}$$

By Theorem 8.1, one has

$$\begin{aligned}
c_n^{G_3}(1) &= \frac{1}{n}\frac{1}{2\pi\sqrt{-1}}\int_C \frac{F_2^n(w_1(z),w_2(z))}{w_2^n(z)} w_1'(z)dz \\
&= \frac{1}{n}\frac{1}{2\pi\sqrt{-1}}\int_C \frac{(w_2(z)+1)^{3n}}{w_2^n(z)} 2(w_2(z)+1)w_2'(z)dz \\
&= \frac{2}{n}\frac{1}{2\pi\sqrt{-1}}\int_C \frac{(w_2+1)^{3n+1}}{w_2^n} dw_2 \\
&= \frac{2}{n}\frac{1}{2\pi\sqrt{-1}}\sum_{k=0}^{3n+1}\binom{3n+1}{k}\int_C w_2^{k-n}dw_2 \\
&= \frac{2}{n}\binom{3n+1}{n-1}.
\end{aligned}$$

Similarly we have

$$\begin{aligned}
c_n^{G_3}(2) &= \frac{1}{n}\frac{1}{2\pi\sqrt{-1}}\int_C \frac{F_2^n(w_1(z),w_2(z))}{w_2^n(z)} w_2'(z)dz \\
&= \frac{1}{n}\frac{1}{2\pi\sqrt{-1}}\int_C \frac{(w_2(z)+1)^{3n}}{w_2^n(z)} w_2'(z)dz \\
&= \frac{1}{n}\frac{1}{2\pi\sqrt{-1}}\int_C \frac{(w_2+1)^{3n}}{w_2^n} dw_2 \\
&= \frac{1}{n}\frac{1}{2\pi\sqrt{-1}}\sum_{k=0}^{3n}\binom{3n}{k}\int_C w_2^{k-n}dw_2 \\
&= \frac{1}{n}\binom{3n}{n-1}.
\end{aligned}$$

Thereofore we obtain

$$c_n^{G_3} = c_n^{G_3}(1) + c_n^{G_3}(2) = \frac{2}{n}\binom{3n+1}{n-1} + \frac{1}{n}\binom{3n}{n-1} = \frac{2}{n+1}\binom{3n}{n}.$$

Hence

$$c_0^{G_3} = 2, \quad c_1^{G_3} = 3, \quad c_2^{G_3} = 10, \quad c_3^{G_3} = 42, \quad c_4^{G_3} = 198, \quad \dots .$$

This sequence is regarded to be the Fibonacci version of the Catalan numbers.

We remark the following equality on the radius R_{G_3} of convergence of the function $f^{G_3}(x)$:

$$\lim_{n\to\infty}\frac{c_n^G}{c_{n+1}^G} = \lim_{n\to\infty}\frac{c_n^G(2)}{c_{n+1}^G(2)} = \lim_{n\to\infty}\frac{(n+1)3n!(2n+3)(2n+2)}{(3n+3)!} = \frac{4}{27}.$$

We note that by the equality $f_1(x) = f_2(x)^2$ one has

$$c_n^{G_3}(1) = \sum_{k=0}^{n} c_k^{G_3}(2)c_{n-k}^{G_3}(2)$$

so that the formula

$$\frac{2}{n}\binom{3n+1}{n-1} = \sum_{k=0}^{n}\frac{1}{k}\binom{3k}{k-1}\frac{1}{n-k}\binom{3n-3k}{n-k-1}$$

holds.

We denote by β the golden ratio $\frac{1+\sqrt{5}}{2}$. It is the Perron-Frobenius eigenvalue of the matrix $A^{G_3} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. The vector $\begin{bmatrix} \beta^{-2} \\ \beta^{-3} \\ \beta^{-2} \end{bmatrix}$ is the unique positive eigenvector for the eigenvalue β whose sum is one. It then folllows that $\varphi(P_1) = \beta^{-1}, \varphi(P_2) = \beta^{-2}$. Therefore we have

$$c_n^{G_3,\varphi}(1) = c_n^{G_3}(1) \cdot \varphi(P_1) = \frac{1}{\beta}\frac{2}{n}\binom{3n+1}{n-1},$$
$$c_n^{G_3,\varphi}(1) = c_n^{G_3}(2) \cdot \varphi(P_2) = \frac{1}{\beta^2}\frac{1}{n}\binom{3n}{n-1}$$

and

$$c_n^{G_3,\varphi} = c_n^{G_3,\varphi}(1) + c_n^{G_3,\varphi}(2) = \frac{2n\beta + (1-n)}{(n+1)(2n+1)}\binom{3n}{n} = \frac{n\sqrt{5}+1}{(n+1)(2n+1)}\binom{3n}{n}.$$

11. Concluding remarks

Let Λ_G be the topological Markov shift of the edges of the graph G as in Section 3. Let $F(A^G)$ be the sub-Fock space associated with the matrix A^G. It is the Hilbert space of the direct sum $F(A^G) = \bigoplus_{n=0}^{\infty} F_n(A^G)$ of the sequence $F_n(A^G)$ of the finite dimensional Hilbert spaces whose orthonomal basis consists of the vectors indexed by the admissible words of the topological Markov shift Λ_G of length n for $n = 1, 2, \dots$. For $n = 0$, the space $F_0(A^G)$ denotes the one dimensional vector space $\mathbb{C}\Omega$ of the vacuum vector Ω. Let $T_e, e \in E$ be the creation operators on $F(A^G)$. By [6] (cf.[8]), the quotient images $\bar{T}_e, e \in E$ by the C^*-algebra of compact operators on $F(A^G)$ satisfy the relations (3.1). We put $T_G = \sum_{e \in E} T_e$. Then the equality $< (T_G + T_G^*)^{2n}\Omega \mid \Omega >= c_n^G$ was pointed out by Yoshimichi Ueda. The author thanks to him for his suggestion. Related discussions are seen in several papers of free probability theory (cf. [5],[11],[29],[30], etc.).

By the above formula of the G-Catalan numbers, one may generalize the numbers c_n^G to general subshifts. For a general subshift Λ over alphabet Σ, let $F(\Lambda)$ be the sub-Fock Hilbert space associated with it ([20]). The creation operators T_α for $\alpha \in \Sigma$ are similarly defined. We put the operator $T_\Lambda = \sum_{\alpha \in \Sigma} T_\alpha$ on $F(\Lambda)$. We may define the Λ-Catalan numbers c_n^Λ by the formula

$$c_n^\Lambda =< (T_\Lambda + T_\Lambda^*)^{2n}\Omega \mid \Omega > .$$

The numbers will be studied in [23]

Acknowledgments: The author would like to thank Toshihiro Hamachi, Wolfgang Krieger and Yoshimichi Ueda for their fruitful suggestions and comments. Yoshimichi Ueda also kindly informed to the author the Land's text book [18] and some references on this subject. The author also thank Hiroaki Yoshida who kindly informed to the author the Deutsch's paper [4].

References

[1] N. Chomsky and M. P. Schützenberger, *The algebraic theory of context-free languages*, Computer programing and formal systems, North-Holland (1963), pp. 118–161

[2] J. Cuntz, *Simple C^*-algebras generated by isometries*, Commun. Math. Phys. **57**(1977), pp. 173–185.

[3] J. Cuntz and W. Krieger, *A class of C^*-algebras and topological Markov chains*, Inventions Math. **56**(1980), pp. 251–268.

[4] E. Deutsch, *Dyck path enumeration*, Discrete Math. **204**(1999), pp. 167–202.

[5] K. Dykema, *On certain free product factors via an extended matrix model*, J. Funct. Anal. **112** (1993) pp. 31–61

[6] M. Enomoto, M. Fujii and Y. Watatani, *Tensor algebras on the sub Fock space associated with $\mathcal{O}_A$*, Math. Japon. **26** (1981) pp. 171–177

[7] M. Enomoto, M. Fujii and Y. Watatani, *KMS states for gauge action on $\mathcal{O}_A$*, Math. Japon. **29** (1984) pp. 607–619

[8] D. Evans, *Gauge actions on $\mathcal{O}_A$*, J. Operator Theory **7** (1982) pp. 79–100

[9] W. Feller, *An Introduction to Probability Theory and Its Applications*, John Wiley & Sons, Inc., New York (1957).

[10] T. Hamachi, K. Inoue and W. Krieger, *Subsystems of finite type and semigroup invariants of subshifts*, preprint.

[11] F. Hiai and D. Petz, *The semicircle Law, Free Random Variables and Entropy*, Mathematical Surveys and Monographs 77, Amer. Math. Soc. (2000).

[12] J. E. Hopcroft and J. D. Ullman, *Introduction to Automata Theory, Languages, and Computation*, Addison-Wesley, Reading (2001).

[13] B. P. Kitchens, *Symbolic dynamics*, Springer-Verlag, Berlin, Heidelberg and New York (1998).

[14] W. Krieger, *On the uniqueness of the equilibrium state*, Math. Systems Theory **8** (1974), pp. 97–104.

[15] W. Krieger, *On a class of algebraic subshifts*, preprint

[16] A. Kumjian, D. Pask, I. Raeburn and J. Renault, *Graphs, groupoids and Cuntz-Krieger algebras*, J. Funct. Anal. **144** (1997) pp. 505–541

[17] A. Kumjian, D. Pask and I. Raeburn, *Cuntz-Krieger algebras of directed graphs*, Pacific J. Math. **184** (1998) pp. 161–174

[18] S. K. Land, *Lectures on generating functions*, Student Mathematical Library 23, Amer. Math. Soc. (2003).

[19] D. Lind and B. Marcus, *An introduction to symbolic dynamics and coding*, Cambridge University Press, Cambridge (1995).

[20] K. Matsumoto *On C^*-algebras associated with subshifts* Internat. J. Math. **8** (1997), pp. 357–374.

[21] K. Matsumoto, *On the simple C^*-algebras arising from Dyck systems*, to appear in J. Operator Theory.

[22] K. Matsumoto, *C^*-algebras arising from Dyck systems of topological Markov chains*, preprint.

[23] K. Matsumoto, *in preparation.*

[24] M. Nivat and J.-F. Perrot, *Une généralisation du monoïde bicyclique*, C. R. Acad. Sc. Paris, **271** (1970) pp. 824–827

[25] J. Riordan, *Combinatrial identities*, Wiley, New York , (1968).

[26] J. Riordan, *An introduction to combinatorial analysis*, Princeton University Press, Prinston, N.J., (1980).

[27] R. P. Stanley, *Enumerative combinatrics I*, Wadsworth & Brooks/Cole Advanced Books & Software, Monterey,CA, (1986).

[28] Y. Ueda and Y. Watatani, *A relation between certain interpolated Cuntz algebras and interpolated free group factors*, Proc. Amer. Math. Soc. **128** (2000) pp. 1397–1404

[29] D. Voiculescu, *Symmetries of some reduced free product C^*-algebras* Operator algebras and their connections with topology and ergodic theory (Buşteni, 1983), Lecture Notes in Math., **1132**, Springer, Berlin, 1985, pp. 556–588,

[30] D. Voiculescu, *Circular and semicircular systems and free product factors* Operator algebras, unitary representations, enveloping algebras, and invariant theory (Paris, 1989), 45–60, Progr. Math., 92, Birkhauser Boston, Boston, MA, 1990

Department of Mathematical Sciences, Yokohama City University, 22-2 Seto, Kanazawa-ku, Yokohama, 236-0027 Japan

E-mail address: `kengo@yokohama-cu.ac.jp`